

A bound on the second Betti number of hyperkähler manifolds of complex dimension six*

Justin Sawon

November, 2015

Abstract

Let M be an irreducible compact hyperkähler manifold of complex dimension six. We prove that the second Betti number of M is at most 23.

1 Introduction

An irreducible hyperkähler manifold is a Riemannian manifold of real dimension $4n$ whose holonomy is equal to $\mathrm{Sp}(n)$. The Riemannian metric will be Kählerian with respect to an S^2 -family of complex structures, so henceforth we will always use the *complex* dimension, $2n$. Beauville [2] and Guan [5] independently proved that the second Betti number of an irreducible compact hyperkähler manifold of dimension four is bounded above by 23. The Hilbert scheme of two points on a K3 surface has second Betti number 23, so this bound is sharp. In this article we prove that the second Betti number of an irreducible compact hyperkähler manifold of dimension six is also bounded above by 23. Up to deformation, there are currently three known examples of such manifolds: the Hilbert scheme of three points on a K3 surface, the generalized Kummer variety (see Beauville [1]), and an example of O’Grady [8]. These examples have second Betti numbers 23, 7, and 8, respectively, so once again our bound is sharp.

Why is it important to bound the second Betti number? The first Pontryagin class $p_1(M)$ determines a homogeneous polynomial of degree $2n-2$ on $H^2(M, \mathbb{Z})$, given by $\alpha \mapsto \int_M \alpha^{2n-2} p_1(M)$. Huybrechts [6] proved that if the second integral cohomology H^2 and the homogeneous polynomial of degree $2n-2$ on H^2 determined by the first Pontryagin class are fixed, then up to diffeomorphism there are only finitely many irreducible compact hyperkähler manifolds of dimension $2n$ realizing this structure. (Instead, one can fix H^2 and a certain normalization \tilde{q} of the Beauville-Bogomolov quadratic form on H^2 and arrive at the same conclusion; see [6].) By bounding the second Betti number, we see that there are finitely many possibilities for H^2 as a \mathbb{Z} -module; it remains to bound the other data on H^2 , to conclude that there are finitely many diffeomorphism types of irreducible compact hyperkähler manifolds of dimension six.

The author would like to thank Nikon Kurnosov for conversations on this work, and the NSF for support (grant number DMS-1206309).

*2010 *Mathematics Subject Classification.* 53C26.

2 Dimension four

Let us recall how to bound the second Betti number in dimension four. Salamon [10] proved that the Betti numbers of a compact hyperkähler manifold of dimension $2n$ satisfy the relation

$$2 \sum_{j=1}^{2n} (-1)^j (3j^2 - n) b_{2n-j} = n b_{2n}.$$

Theorem 1 (Beauville [2], Guan [5]) *Let M be an irreducible compact hyperkähler manifold of complex dimension four. Then the second Betti number b_2 of M is at most 23.*

Proof Irreducible hyperkähler manifolds are simply-connected, so $b_1 = 0$. Therefore Salamon's relation for $n = 2$ gives

$$-2b_3 + 20b_2 + 92 = 2b_4.$$

Verbitsky [11] proved that $\text{Sym}^k H^2(M, \mathbb{R})$ injects into $H^{2k}(M, \mathbb{R})$ for $k \leq n$. In particular, we can write

$$H^4(M, \mathbb{R}) \cong \text{Sym}^2 H^2(M, \mathbb{R}) \oplus H_{\text{prim}}^4(M, \mathbb{R})$$

and

$$b_4 = \binom{b_2 + 1}{2} + b'_4,$$

where b'_4 denotes the dimension of the primitive cohomology $H_{\text{prim}}^4(M, \mathbb{R})$. Substituting this into Salamon's relation gives

$$-2b_3 + 20b_2 + 92 = b_2(b_2 + 1) + 2b'_4,$$

and therefore

$$-(b_2 + 4)(b_2 - 23) = -b_2^2 + 19b_2 + 92 = 2b'_4 + 2b_3.$$

The left-hand side is negative if $b_2 > 23$, whereas the right-hand side is clearly non-negative. Therefore the second Betti number b_2 can be at most 23. \square

Example Up to deformation, there are two known examples of irreducible compact hyperkähler manifolds of dimension four: the Hilbert scheme $\text{Hilb}^2 S$ of two points on a K3 surface S (see Fujiki [3]) and the generalized Kummer variety $K_2(A)$ of an abelian surface A (see Beauville [1]). Their Hodge diamonds are

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ & 1 & 21 & 1 & \\ 0 & 0 & & 0 & 0 \\ 1 & 21 & 232 & 21 & 1 \\ 0 & 0 & & 0 & 0 \\ & 1 & 21 & 1 & \\ & 0 & & 0 & \\ & & 1 & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ & 1 & 5 & 1 & \\ 0 & 4 & & 4 & 0 \\ 1 & 5 & 96 & 5 & 1 \\ 0 & 4 & & 4 & 0 \\ & 1 & 5 & 1 & \\ & 0 & & 0 & \\ & & 1 & & \end{array},$$

with $b_2 = 23$, $b_3 = 0$, $b'_4 = 0$, and $b_2 = 7$, $b_3 = 8$, $b'_4 = 80$, respectively. In fact, if $b_2 = 23$ then b_3 and b'_4 must both vanish.

3 Dimension six

In higher dimensions, the injection $\text{Sym}^k H^2(M, \mathbb{R}) \hookrightarrow H^{2k}(M, \mathbb{R})$ is insufficient to produce a bound on the second Betti number. Instead we employ the following refinement.

Theorem 2 (Verbitsky [12], Looijenga and Lunts [7]) *Let M be an irreducible compact hyperkähler manifold of dimension $2n$ with second Betti number b_2 . Then there is an action of $\mathfrak{so}(4, b_2 - 2)$ on the real cohomology $\bigoplus_{k=0}^{4n} H^k(M, \mathbb{R})$, and hence of $\mathfrak{so}(b_2 + 2, \mathbb{C})$ on the complex cohomology $\bigoplus_{k=0}^{4n} H^k(M, \mathbb{C})$.*

Remark This action is generated by Lefschetz operators: for each Kähler class $[\omega]$ the operators $L_{[\omega]}$ and $\Lambda_{[\omega]}$ generate an $\mathfrak{sl}(2, \mathbb{C})$ -action on the complex cohomology, and the amalgamation of all these actions yields the $\mathfrak{so}(b_2 + 2, \mathbb{C})$ -action.

We can decompose $\bigoplus_{k=0}^{4n} H^k(M, \mathbb{C})$ into irreducible representations for this $\mathfrak{so}(b_2 + 2, \mathbb{C})$ -action. Their highest weights are related to Hodge bi-degrees; indeed, the Hodge diamond is the projection onto a plane of the (higher-dimensional) weight lattice of $\mathfrak{so}(b_2 + 2, \mathbb{C})$. We can choose positive roots so that the dominant Weyl chamber projects onto the shaded octant of the Hodge diamond shown in Figure 1.

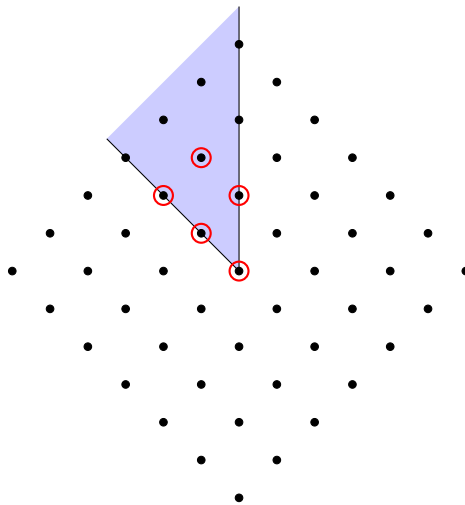


Figure 1: The Hodge diamond in dimension six

The irreducible representation with highest weight vector $1 \in H^0(M, \mathbb{C})$ is precisely the subring of the cohomology generated by $H^2(M, \mathbb{C})$. In dimension six, the remainder of the cohomology comes from irreducible representations V_1, V_2, V_3, V_4 , and V_5 whose highest weight vectors lie in the Hodge groups that are circled in Figure 1 and described in Table 1. The dimensions of the spin representations are not needed for our arguments.

	highest weight vector in	$\mathfrak{so}(b_2 + 2, \mathbb{C})$ -module	dimension
V_1	$H^{2,1}(M)$	a spin representation	
V_2	$H^{3,1}(M)$	$\Lambda^2 \mathbb{C}^{b_2+2}$	$(b_2 + 2)(b_2 + 1)/2$
V_3	$H^{2,2}(M)$	\mathbb{C}^{b_2+2}	$b_2 + 2$
V_4	$H^{3,2}(M)$	a spin representation	
V_5	$H^{3,3}(M)$	trivial	1

Table 1: Irreducible representations of $\mathfrak{so}(b_2 + 2, \mathbb{C})$ occurring in the cohomology of M

Moreover, V_2 will sit inside the Hodge diamond in the following manner (where we have indicated the dimension of $V_2^{p,q}$ for each p, q)

$$\begin{array}{cccccccc}
& & & & 0 & & & \\
& & & & 0 & & 0 & \\
& & & 0 & & 0 & & 0 \\
& & 0 & & 0 & & 0 & 0 \\
& 0 & & 1 & & b_2 - 2 & & 1 & 0 \\
& 0 & 0 & & 0 & & 0 & & 0 & 0 \\
0 & 0 & & b_2 - 2 & & \frac{b_2^2 - 5b_2 + 10}{2} & & b_2 - 2 & 0 & 0 & 0 \\
& 0 & 0 & & 0 & & 0 & & 0 & & 0 \\
& & 0 & & 1 & & b_2 - 2 & & 1 & & 0 \\
& & & 0 & & 0 & & 0 & & 0 & 0 \\
& & & & 0 & & 0 & & 0 & & 0 \\
& & & & & 0 & & 0 & & 0 & 0 \\
& & & & & & 0 & & 0 & & 0
\end{array}$$

whereas V_3 will sit inside the Hodge diamond as

$$\begin{array}{cccccccc}
& & & & 0 & & & \\
& & & & 0 & & 0 & \\
& & & & 0 & & 0 & 0 \\
& & & 0 & & 0 & & 0 \\
& & 0 & & 0 & & 1 & & 0 & 0 \\
& 0 & & 0 & & 0 & & 0 & & 0 & 0 \\
0 & 0 & & 1 & & b_2 - 2 & & 1 & & 0 & 0 & 0 \\
& 0 & 0 & & 0 & & 0 & & 0 & & 0 & 0 \\
& & 0 & & 0 & & 1 & & 0 & & 0 & 0 \\
& & & 0 & & 0 & & 0 & & 0 & 0 & 0 \\
& & & & 0 & & 0 & & 0 & & 0 & 0 \\
& & & & & 0 & & 0 & & 0 & & 0 \\
& & & & & & 0 & & 0 & & 0 & 0
\end{array}$$

With these preliminaries out of the way, we can prove our main result.

Theorem 3 *Let M be an irreducible compact hyperkähler manifold of complex dimension six. Then the second Betti number b_2 of M is at most 23.*

Proof When $n = 3$ Salamon's relation gives

$$18b_4 - 48b_3 + 90b_2 + 210 = 3b_6.$$

with $b_2 = 23$, $b_3 = 0$, $c = 1$, $d = 0 = e = 0$, and $b_2 = 7$, $b_3 = 8$, $c = 1$, $d = 16$, $e = 240$, respectively. The Hodge numbers of O'Grady's example M_6 were calculated by Mongardi, Rapagnetta, and Saccà [9]; they are

with $b_2 = 8$, $b_3 = 0$, $c = 6$, $d = 115$, $e = 290$.

When $n = 4$ Salamon's relation gives

$$2b_7 + 16b_6 - 46b_5 + 88b_4 - 142b_3 + 208b_2 + 376 = 4b_8.$$

Thus in dimension eight, b_7 appears with a coefficient of the ‘wrong’ sign, and we cannot simply imitate the proof of Theorem 3. We can prove the following weaker result.

Theorem 4 *Let M be an irreducible compact hyperkähler manifold of complex dimension eight whose odd Betti numbers all vanish. Then the second Betti number b_2 of M is at most 24.*

Proof The proof is essentially the same as that of Theorem 3 so we omit most of the details. After decomposing the complex cohomology of M into irreducible representations of $\mathfrak{so}(b_2 + 2, \mathbb{C})$, we obtain formulae for b_4 , b_6 , and b_8 in terms of b_2 and certain non-negative multiplicities c, d, e, f, \dots Substituting these into Salamon's relation gives

$$-(b_2 + 3)(b_2 + 8) \left(b_2 - \frac{21 + \sqrt{817}}{2} \right) \left(b_2 - \frac{21 - \sqrt{817}}{2} \right) = R.H.S.$$

The left-hand side is negative if $b_2 \geq 25 > \frac{21+\sqrt{817}}{2} \sim 24.7916$, whereas for $b_2 \geq 25$ the right-hand side will be a non-negative linear combination of the non-negative multiplicities c, d, e, f, \dots . Therefore the second Betti number b_2 can be at most 24. \square

Remark We could have assumed that only b_7 vanishes, but because of the $\mathfrak{so}(b_2 + 2, \mathbb{C})$ -action on the cohomology this would already imply that *all* odd Betti numbers vanish.

Remark The pattern appears to be that in dimension $2n$, the polynomial in b_2 on the left-hand side has largest root $\frac{21+\sqrt{433+96n}}{2}$, so that an irreducible compact hyperkähler manifold whose odd Betti numbers all vanish must have second Betti number $b_2 \leq \frac{21+\sqrt{433+96n}}{2}$. The author has not rigorously verified this.

[1] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), no. 4, 755–782.

- [2] A. Beauville, private communication, 1999.
- [3] A. Fujiki, *On primitively symplectic compact Kähler V-manifolds of dimension four*, in Classification of algebraic and analytic manifolds (Katata, 1982), 71–250, Progr. Math. **39**, Birkhäuser Boston, Boston, MA, 1983.
- [4] L. Götsche and W. Soergel, *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, Math. Ann. **296** (1993), no. 2, 235–245.
- [5] D. Guan, *On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four*, Math. Res. Lett. **8** (2001), no. 5–6, 663–669.
- [6] D. Huybrechts, *Finiteness results for compact hyperkähler manifolds*, J. Reine Angew. Math. **558** (2003), 15–22.
- [7] E. Looijenga and V. Lunts, *A Lie algebra attached to a projective variety*, Invent. Math. **129** (1997), no. 2, 361–412.
- [8] K. O’Grady, *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. **12** (2003), no. 3, 435–505.
- [9] G. Saccà, private communication, 2015.
- [10] S. Salamon, *On the cohomology of Kähler and hyper-Kähler manifolds*, Topology **35** (1996), no. 1, 137–155.
- [11] M. Verbitsky, *Actions of the Lie algebra of $SO(5)$ on the cohomology of a hyper-Kähler manifold*, (English translation) Funct. Anal. Appl. **24** (1990), no. 3, 229–230.
- [12] M. Verbitsky, *Cohomology of compact hyper-Kähler manifolds and its applications*, Geom. Funct. Anal. **6** (1996), no. 4, 601–611.

Department of Mathematics
University of North Carolina
Chapel Hill NC 27599-3250
USA

sawon@email.unc.edu
www.unc.edu/~sawon